On weak approximation of stochastic differential equations with discontinuous drift coefficient

(joint work with A. Kohatsu-Higa and A. Lejay)

Kazuhiro YASUDA

Hosei University

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Introduction 1 (Problem)

- In this talk we consider the following d-dimensional SDE:

$$dX_t = \mathbf{b(t, X_t)}dt + \sigma(t, X_t)dW_t. \tag{1}$$

  where $b(t,x)$, $\sigma(t,x)$ are suitable functions and W_t is a Brownian motion.
- We would like to consider a **weak convergence rate of (1) with a discontinuous drift coefficient $b(t,x)$**.
- For simplicity, we split the interval $[0, T]$ equally in n subintervals and let the length of each time subinterval Δt be equal to $\frac{T}{n}$.
- We say that an approximation process $\bar{X}_T$ weakly converges to X_T with order γ if

$$\left| E\left[f\left(X_T\right)\right] - E\left[f\left(\bar{X}_T\right)\right] \right| \leq C\Delta t^{\gamma}.$$

  holds for every f in a certain class.

Introduction 2 (Previous studies: continuous coeff.)

- When we use the Euler-Maruyama approximation as the approximation process $\bar{X}_T$, then the following result has been known.
- For $\alpha \in (0,1) \cup (1,2) \cup (2,3)$, let $H_T^{(\alpha)}$ be the Hölder space on $[0,T] \times \mathbb{R}^d$ and $H^{(\alpha)}$ be also the Hölder space on $\mathbb{R}^d$.
- If $b,\ \sigma\sigma^* \in H_T^{(\alpha)}$ **and** $f \in H^{(2+\alpha)}$ for some $\alpha \in (0,1) \cup (1,2) \cup (2,3)$, there exists some positive constant K such that

$$\left| E\left[f\left(X_T\right)\right] - E\left[f\left(\bar{X}_T\right)\right]\right| \leq \frac{K}{n^{E(\alpha)}},$$

where

$$E(\alpha) = \begin{cases} \frac{\alpha}{2}, & \alpha \in (0,1), \quad \leftarrow \quad 0 \sim \frac{1}{2} \\ \frac{1}{3-\alpha}, & \alpha \in (1,2), \quad \leftarrow \quad \frac{1}{2} \sim 1 \\ 1, & \alpha \in (2,3). \end{cases}$$

- For more details, see Mikulevicius and Platen [5] or a book of Kloeden and Platen [3].

Introduction 3 (Previous studies: **dis**continuous coeff.)

- In this talk, we are interested in a discontinuous drift coefficient $b(t,x)$.
- Some results of weak convergence of SDE with discontinuous (drift and **diffusion**) coefficient and the Euler-Maruyama approximation:
  - Chan, Stramer (1998) [1]:

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t).$$

    If $b,\ \sigma$ **are piecewise continuous** and locally bounded, then the Euler-Maruyama approximation weakly converges. Note that they do not mention about the rate.
  - Yan (2002) [9]:

$$dX(t) = b(t,X(t))dt + \sigma(t,X(t))dW(t).$$

    If $b,\ \sigma$ **have that the set of all discontinuous points has measure** 0 and linear growth, then the Euler-Maruyama approximation weakly converges. Note that they do not mention about the rate, neither.

Contents

Settings and Assumptions 1 (Original SDE)

- Let a fixed $T > 0$.
- $\sigma(t,x)$: $d \times d$-symmetric matrix value **uniformly continuous function** on $[0,T] \times \mathbb{R}^d$, and there exist some positive constants $\Lambda \geq \lambda > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$,

$$\lambda|\xi|^2 \leq \xi^* a(t,x)\xi \leq \Lambda|\xi|^2,$$

  where set $a(t,x) = \sigma\sigma^*(t,x)$.
- $b(t,x)$: d-dimensional **measurable function** on $[0,T] \times \mathbb{R}^d$, and for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $|b(t,x)| \leq \Lambda$ holds.
- Then the SDE (1) has a weak solution.
- $C_{SI}(\mathbb{R}^d)$: a class of all **continuous function f** such that for all $k > 0$,

$$\lim_{|x|\to\infty} |f(x)|e^{-k|x|^2} = 0.$$

Settings and Assumptions 2 (Approximation)

- Let $\varepsilon > 0$.
- $b_\varepsilon(t,x)$: d-dimensional measurable function on $[0,T]\times\mathbb{R}^d$, and for all $(t,x)\in[0,T]\times\mathbb{R}^d$, $|b(t,x)|\le\Lambda$ holds.
  $\Longrightarrow$ Later we assume Hölder continuity or smoothness as necessary.
- Then we consider the following SDE:

$$dX_t^\varepsilon = x + \int_0^t b_\varepsilon(s, X_s^\varepsilon)ds + \int_0^t \sigma(s, X_s^\varepsilon)\,dB_s.$$

  The drift coefficient $b(t,x)$ is replaced by $b_\varepsilon(t,x)$.
- We consider the Euler-Maruyama approximation of X_t^ε with $\Delta t = \frac{T}{n}$:

$$\bar{X}_t^\varepsilon = x + \int_0^t b_\varepsilon\left(\phi(s), \bar{X}_{\phi(s)}^\varepsilon\right)ds + \int_0^t \sigma\left(\phi(s), \bar{X}_{\phi(s)}^\varepsilon\right)dB_s,$$

  where $\phi(s) = \sup\{t\le s | t = \frac{k}{n} \text{ for } k\in\mathbb{N}\}$.

Theorem 1

Theorem

We assume that for $\gamma, \beta, \delta > 0$, the following two inequalities hold:

(i). *for* $\gamma > 0$, $\left|E\left[f\left(X_T\right)\right] - E\left[f\left(X_T^{\varepsilon}\right)\right]\right| = O\left(\varepsilon^{\gamma}\right)$,

(ii). *and for* $\beta,\ \delta > 0$, $\left|E\left[f\left(X_T^{\varepsilon}\right)\right] - E\left[f\left(\bar{X}_T^{\varepsilon}\right)\right]\right| = O\left(\frac{1}{\varepsilon^{\beta} n^{\delta}}\right)$.

Then for $\varepsilon = O(n^{-\frac{\delta}{\gamma+\beta}})$, *the following holds*：

$$\left|E\left[f(X_T)\right] - E\left[f\left(\bar{X}_T^{\varepsilon}\right)\right]\right| \leq O\left(n^{-\frac{\delta\gamma}{\gamma+\beta}}\right).$$

Remark:

- Note that $\bar{X}_T^{\varepsilon}$ is **NOT** the direct Euler-Maruyama approximation of X_T.
- When $\varepsilon = n^{-\frac{\delta}{\gamma+\beta}}$, the following holds：

$$\frac{1}{\varepsilon^{\beta} n^{\delta}} = n^{-\frac{\delta\gamma}{\gamma+\beta}} \overset{n\to\infty}{\longrightarrow} 0.$$

Proposition 1 (About assumption **(i)**)

On the assumption **(i)** in Theorem:

Proposition

For some $\alpha, p > 2$ such that $\frac{1}{\alpha} + \frac{1}{p} < \frac{1}{2}$, and all $f \in C_{SI}(\mathbb{R}^d)$, we have

$$\left| E\left[f(X_T)\right] - E\left[f\left(X_T^\varepsilon\right)\right]\right| \leq C(\alpha, p, T) A_T(\varepsilon) \sqrt{Var(f(X_T))},$$

where set

$$C(\alpha, p, T) = T^{\frac{1}{2}-\frac{1}{p}} \exp\left(T\Lambda\lambda^{-1}\left(\alpha - \frac{1}{2} + \left(1 - \frac{2}{\alpha}\right)\frac{\alpha(\frac{1}{2} + \frac{1}{p}) - 1}{\alpha(\frac{1}{2} - \frac{1}{p}) - 1}\right)\right),$$

$$A_T(\varepsilon) = E\left[\int_0^T \left|b_\varepsilon(s, Y_s) - b(s, Y_s)\right|^p ds\right]^{\frac{1}{p}},$$

where Y_t is a weak solution of $Y_t = x + \int_0^t \sigma(s, Y_s) dW_s$.

Remark 1 (About $A_T(\varepsilon)$)

Remark

- *If the transition density function* $p(t,x,y)$ *of* Y_t *has a Gaussian upper estimation, then for every* $1 < r, q \leq \infty$ *such that* $\frac{d}{2r} + \frac{1}{q} < 1$, *we have*

$$A_T(\varepsilon) \leq C_3 \left\{ \int_0^T \left(\int_{\mathbb{R}^d} \left| b(s,y) - b_\varepsilon(s,y) \right|^{pq} dy \right)^{\frac{r}{q}} ds \right\}^{\frac{1}{rp}}.$$

  A rate of convergence depends on a kind of L^p*-error between* b *and* b_ε. *When* $a = \sigma\sigma^* \in H^{\frac{\alpha}{2},\alpha}([0,T] \times \mathbb{R}^d)$ $(\alpha > 0)$ *holds, it has a Gaussian upper estimation.*

- *Even if it does NOT have a Gaussian upper estimation, from bounded and uniformly elliptic assumptions, we have the following when* $b(t,x) = b(x)$,

$$A_T(\varepsilon) \leq C(\lambda, \Lambda) e^T \|b - b_\varepsilon\|_{L^{dp}}.$$

Proposition 2 (About assumption **(ii)**)

On the assumption **(ii)** in Theorem:

- When we adopt a **Hölder continuous function** as b_ε, we can use the following result (which was mentioned before). For example, a broken line approximation is in this case.

Proposition

(Mikulevicius and Platen [5]) If $b_\varepsilon,\ \sigma\sigma^* \in H_T^{(\alpha)}$ **and** $f \in H^{(2+\alpha)}$ *for some* $\alpha \in (0,1)\cup(1,2)\cup(2,3)$*, there exists some positive constant* K *such that*

$$\left| E\left[f\left(X_T^\varepsilon\right)\right] - E\left[f\left(\bar{X}_T^\varepsilon\right)\right]\right| \leq \frac{K}{n^{E(\alpha)}},$$

where

$$E(\alpha) = \begin{cases} \frac{\alpha}{2}, & \alpha \in (0,1), \quad \leftarrow\ 0 \sim \frac{1}{2} \\ \frac{1}{3-\alpha}, & \alpha \in (1,2), \quad \leftarrow\ \frac{1}{2} \sim 1 \\ 1, & \alpha \in (2,3). \end{cases}$$

- Note that the constant K **linearly** depends on $\|b_\varepsilon\|_{H^\alpha}$.

Proposition 3 (About assumption **(ii)**)

- When b is quite complicate and we use a mollifier (**smooth approximation**) as b_ε, then

Proposition

Let $f \in C^3(\mathbb{R}^d) \cap C_{SI}(\mathbb{R}^d)$ **and** $b_\varepsilon,\ \sigma \in C_b^{1,3}([0,T] \times \mathbb{R}^d)$. *Then we have*

$$\left| E\left[f\left(X_T^\varepsilon \right) \right] - E\left[f\left(\bar{X}_T^\varepsilon \right) \right] \right| \leq \frac{C}{n} ||b_\varepsilon||_{3,\infty},$$

where C *is a positive constant and* $||b_\varepsilon||_{3,\infty}$ *is defined as follows:*

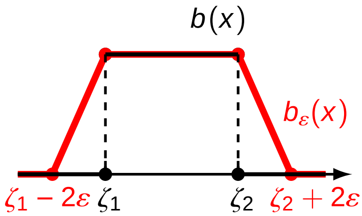
$$||b_\varepsilon||_{3,\infty} = \sum_{j=0}^{3} \left\| b_\varepsilon^{(j)} \right\|_\infty .$$

Example (Indicator function)

Remark

- *Set* $d = 1$, $b(t,x) = \mathbf{1}_{[\zeta_1,\zeta_2]}(x)$ $(\zeta_1 < \zeta_2)$ *and*

$$b_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon}x - \frac{\zeta_1 - 2\varepsilon}{2\varepsilon}, & [\zeta_1 - 2\varepsilon, \zeta_1), \\ -\frac{1}{2\varepsilon}x + \frac{\zeta_2 + 2\varepsilon}{2\varepsilon}, & (\zeta_2, \zeta_2 + 2\varepsilon], \\ 1, & [\zeta_1, \zeta_2], \\ 0, & \textit{Otherwise}. \end{cases}$$



(broken line approximation)

- *Assumption* **(i)**: *For* $p > 2$, *we have*

$$\left\{ \int_{-\infty}^{\infty} |b_\varepsilon(x) - b(x)|^p dx \right\}^{\frac{1}{p}} = O(\varepsilon^{\frac{1}{p}}).$$

- *Assumption* **(ii)**: *The rate of the divergence is* $\|b_\varepsilon\|_{H^\alpha} = O(\varepsilon^{-1})$.
- *An optimal rate of* ε *is* $\varepsilon = O(n^{-\frac{p}{2(1+p)}})$.

Settings and Assumptions 3 (Constant diffusion coeff.)

- Now we assume that $\sigma(t,x)$ is a constant matrix (Here unit-matrix) and $b(t,x) = b(x)$ is time-homogeneous.
  That is, we consider the following SDE:

$$X_t = x + \int_0^t b(X_s)ds + B_t.$$

- X_t^ε : The solution of SDE with an approximated drift $b_\varepsilon(x)$.
- $\bar{X}_t^\varepsilon$: The Euler-Maruyama approximation of X_t^ε.
- The Euler-Maruyama approximation of X_t :

$$\bar{X}_t = x + \int_0^t b\left(\bar{X}_{\phi(s)}\right)ds + B_t.$$

- Until the previous slide, we consider about weak approximation between X_t and $\bar{X}_t^\varepsilon$ which is **NOT** the direct Euler-Maruyama approximation of X_t. Now as the approximation $\bar{X}_t$ of X_t, we consider the direct Euler-Maruyama approximation of X_t.

Lemmas

Lemma

For $p > 2 \vee d$, there exists some positive constant $C_1(p, \Lambda, T)$ such that

$$\left| E\left[f\left(X_T\right)\right] - E\left[f\left(X_T^{\varepsilon}\right)\right]\right| \le C_1(p, \Lambda, T)\sqrt{\mathrm{Var}(f(x + B_T))}\|\mathbf{b} - \mathbf{b}_{\varepsilon}\|_{\mathbf{L^p}}.$$

Lemma

For $p > 2$, there exists some positive constant $C_2(p, \Lambda, T)$ such that

$$\left| E\left[f\left(\bar{X}_T\right)\right] - E\left[f\left(\bar{X}_T^{\varepsilon}\right)\right]\right| \le C_2(p, \Lambda, T)\sqrt{\mathrm{Var}(f(x + B_T))}\|\mathbf{b} - \mathbf{b}_{\varepsilon}\|_{\mathbf{L^p}}.$$

- The both lemmas are similar results to Proposition 1.
- We use a happy property of the Euler-Maruyama approximation in the case of the constant diffusion coefficient: $\sum_{i=1}^{n}(B_{i\Delta t} - B_{(i-1)\Delta t}) = B_T$.

Proposition 4

Proposition

(Theorem 1 in Mackevičius [4]) Let b_ε be a bounded and Lipschitz continuous function with constant $\mathrm{Lip}(b_\varepsilon)$ and f be in $C_p^3(\mathbb{R}^d)$. Then there exists some positive constant $C(T,\Lambda,f)$ such that

$$\left|E\left[f\left(X_T^\varepsilon\right)\right] - E\left[f\left(\bar{X}_T^\varepsilon\right)\right]\right| \leq \frac{C(T,\Lambda,f)}{n}\mathrm{Lip}(b_\varepsilon).$$

- In the paper of Mackevičius [4], the boundedness of b_ε is not assumed.

Notations:

- For a set G in $\mathbb{R}^d$, we define $G(\varepsilon) = \{x \in \mathbb{R}^d | d(x,G) \leq \varepsilon\}$, where $d(x,G) = \inf_{y\in G}|x-y|$ is the distance between x and G.

By summing up the above results, we have the following theorem.

Theorem 2

Theorem

Let b be a bounded measurable function on $\mathbb{R}^d$ which is Lipschitz except on a set G such that the Lebesgue measure $\mathrm{meas}(G(\varepsilon)) = O(\varepsilon^d)$. Then for any $f \in C_p^3(\mathbb{R}^d)$ and $p > d \vee 2$, we have

$$\left|E\left[f(X_T)\right] - E\left[f\left(\bar{X}_T\right)\right]\right| = O\left(n^{-\frac{d}{p+d}}\right).$$

Remark

- *An optimal size of ε is $\varepsilon = O(n^{-\frac{p}{d+p}})$, where we assume $\mathrm{Lip}(b_\varepsilon) = O(\varepsilon^{-1})$ and $\|b - b_\varepsilon\|_{L^p} = O(\varepsilon^{\frac{d}{p}})$.*
- *From the theorem, we find that if $p \to 2$ and $d = 1$, the rate of the weak convergence is* **order** $\frac{1}{3}$*. Note that for $\alpha \in (0, 1)$ (Hölder continuous case), the result in Kloeden and Platen [3] is the* **order** $\frac{1}{2}$*. (where in their case, σ is also α–Hölder continuous.)*

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Numerical Experiments 1

- We consider the following SDE ($d = 1$):

$$X_t = x + \int_0^t b(X_s)ds + B_t, \quad \text{where } b(x) = \begin{cases} \theta_1, & x \le 0, \\ \theta_0, & x > 0. \end{cases}$$

- From Karatzas and Shreve [2], we have a representation of the transition density function of X_t.
- If $\theta_1 = -\theta_0 > 0$ and $x = 0$, the distribution of X_t is symmetric. And if f is an odd function, we have $E[f(X_t)] = 0$.
- As b_ε, we use

$$b_\varepsilon(x) = \begin{cases} \theta_1, & x \le -\varepsilon, \\ \frac{\theta_0 - \theta_1}{2\varepsilon} x + \frac{\theta_0 + \theta_1}{2}, & -\varepsilon < x \le \varepsilon, \\ \theta_0, & x > \varepsilon. \end{cases}$$

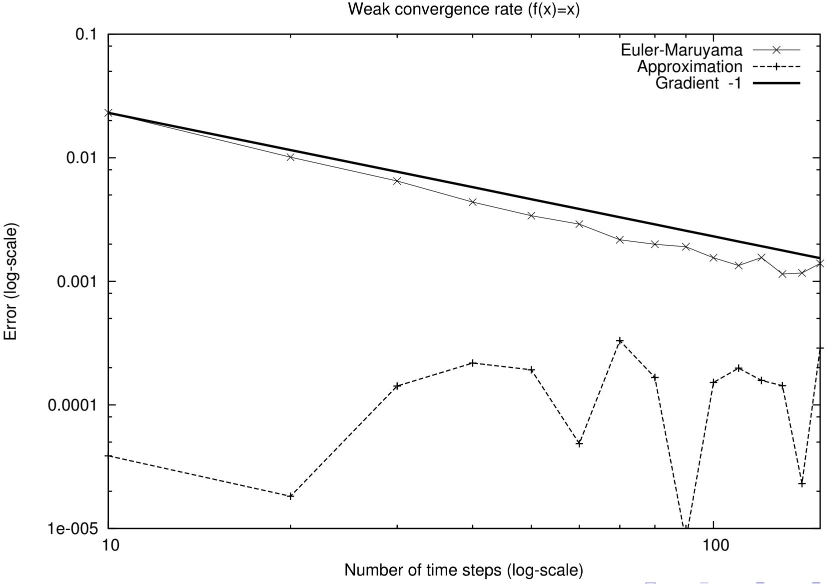
- Here we adopt $\varepsilon = n^{-\frac{2}{3}}$ from the previous slide.

Numerical Experiments 2

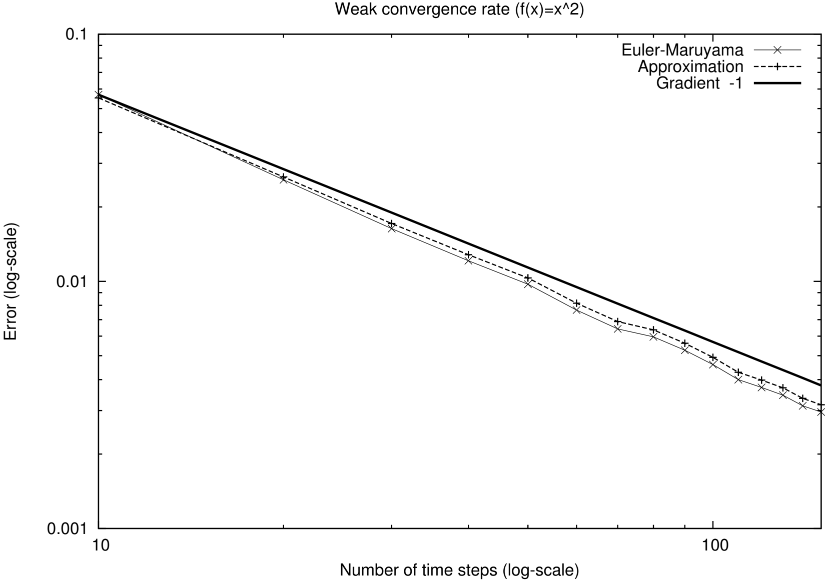
- We consider two types of errors:
  - $|E[f(X_T)] - E[f(\bar{X}_T)]|$: thin line in all graphs below.
  - $|E[f(X_T)] - E[f(\bar{X}_T^{\varepsilon})]|$: dotted line in all graphs below.
- Let $T = 1$, $\theta_1 = -\theta_0 = 1$ and $X_0 = 0$.
- We use 10^7 times Monte-Carlo simulations to $E[f(\bar{X}_1)]$ and $E[f(\bar{X}_1^{\varepsilon})]$ each time step n.

- First Example: $f(x) = x$.
  - True value: $E[f(X_1)] = 0$ from the odd function f and the initial value $X_0 = 0$.
- Second Example: $f(x) = x^2$.
  - True value: $E[f(X_1)] = 0.333369$ which is analytically obtained.
- Third Example: $f(x) = \mathbf{1}(x > 0) - \mathbf{1}(x \leq 0)$. (This is outside of our theorem.)
  - True value: $E[f(X_1)] = 0$ from the odd function f a.e. and the initial value $X_0 = 0$.

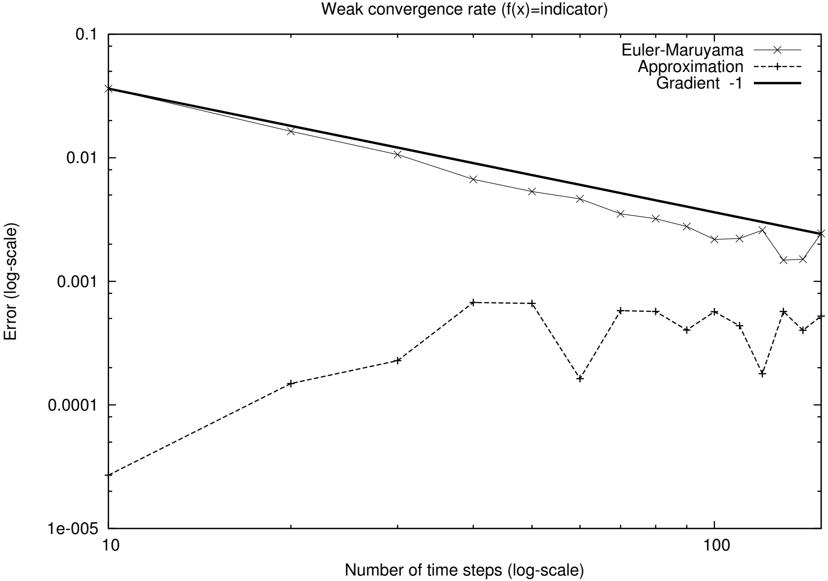
Numerical Experiments 3: $f(x) = x$



Numerical Experiments 4: $f(x) = x^2$



Numerical Experiments 5: $f(x) = \mathbf{1}(x > 0) - \mathbf{1}(x \leq 0)$



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Proposition 1

On the assumption **(i)** in Theorem:

Proposition

For some $\alpha, p > 2$ such that $\frac{1}{\alpha} + \frac{1}{p} < \frac{1}{2}$, and all $f \in C_{SI}(\mathbb{R}^d)$, we have

$$\left| E\left[f(X_T)\right] - E\left[f\left(X_T^{\varepsilon}\right)\right]\right| \leq C(\alpha, p, T) A_T(\varepsilon) \sqrt{Var(f(X_T))},$$

where set

$$C(\alpha, p, T) = T^{\frac{1}{2}-\frac{1}{p}} \exp\left(T\Lambda\lambda^{-1}\left(\alpha - \frac{1}{2} + \left(1 - \frac{2}{\alpha}\right)\frac{\alpha(\frac{1}{2}+\frac{1}{p}) - 1}{\alpha(\frac{1}{2}-\frac{1}{p}) - 1}\right)\right),$$

$$A_T(\varepsilon) = E\left[\int_0^T \left|b_\varepsilon(s, Y_s) - b(s, Y_s)\right|^p ds\right]^{\frac{1}{p}},$$

where Y_t is a weak solution of $Y_t = x + \int_0^t \sigma(s, Y_s) dW_s$.

Proof of Proposition 1 (1)

- We define Z_t as

$$Z_t = \exp\left(\int_0^t \gamma(s, X_s)dB_s - \frac{1}{2}\int_0^t \gamma\gamma^*(s, X_s)ds\right),$$

where set $\gamma(s,x) = (b^\varepsilon(s,x) - b(s,x))^*\sigma^{-1}(s,x)$.

Lemma

Set $\hat{\gamma} = \sup_{s\in[0,T],x\in\mathbb{R}^d} |\gamma(s,x)|$. *For* $\alpha > 1$, *we have*

$$E\left[Z_T^\alpha\right]^{\frac{1}{\alpha}} \le \exp\left(\left(\alpha - \frac{1}{2}\right)\hat{\gamma}^2 T\right). \tag{2}$$

Proof. Set $M_t = \int_0^t \gamma(s, X_s)dB_s$. From the Schwarz inequality,

$$E\left[Z_T^\alpha\right] \le E\left[\exp\left(2\alpha M_T - \frac{4\alpha^2}{2} < M >_T\right)\right]^{\frac{1}{2}} E\left[\exp\left((2\alpha^2 - \alpha) < M >_T\right)\right]^{\frac{1}{2}}.$$

We obtain the consequence from the exponential martingale and $< M >_T \le \hat{\gamma}^2 T$. □

Proof of Proposition 1 (2)

- Set $L = \sum_{i,j=1}^{d} \frac{1}{2} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}$.
- We consider the following Cauchy problem:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + Lu(t,x) = 0, & on\ [0,T) \times \mathbb{R}^d, \\ u(T,x) = f(x), & on\ \mathbb{R}^d. \end{cases}$$

Lemma

(Lemma 1 and Corollary, Veretennikov [8]) We have the following representation and estimation by using the solution u:

$$\text{(i).} \quad f(X_T) = u(0,X_0) + \int_0^T \nabla u(s,X_s)\sigma(s,X_s)dB_s,$$

$$\text{(ii).} \quad E\left[\int_0^T |\nabla u(s,X_s)|^2 ds\right] \le \mathrm{Var}(f(X_T)).$$

- About existence and uniqueness, see Theorem 3 and 3′ in [8].

Proof of Proposition 1 (3)

- From Girsanov theorem、 $E\left[f\left(X_T^\varepsilon\right)\right]=E\left[Z_T f(X_T)\right]$.
- Z_t follows to $Z_t=1+\int_0^t Z_s\gamma(s,X_s)dB_s$.
- From the previous Lemma (i) and martingale property of Z_t,

$$
\begin{aligned}
|\Delta| &= \left|E\left[f\left(X_T^\varepsilon\right)\right]-E\left[f\left(X_T\right)\right]\right| = \left|E\left[\left(Z_T-1\right)f\left(X_T\right)\right]\right| \\
&= \left|E\left[Z_T\int_0^T\left(b^\varepsilon(s,X_s)-b(s,X_s)\right)\nabla u(s,X_s)ds\right]\right| \\
&\le \underbrace{E\left[Z_T^\alpha\right]^{\frac{1}{\alpha}}}_{Lemma} E\left[\left(\int_0^T|(b^\varepsilon-b)(s,X_s)|^2ds\right)^{\frac{\alpha'}{2-\alpha'}}\right]^{\frac{2-\alpha'}{2\alpha'}} \underbrace{E\left[\int_0^T|\nabla u(s,X_s)|^2ds\right]^{\frac{1}{2}}}_{Lemma(ii)}.
\end{aligned}
$$

- We use Girsanov theorem to the middle term and similar arguments again. Then we obtain the term related to $A_T(\varepsilon)$. □

Proposition 3

Proposition

Let $f \in C^3(\mathbb{R}^d) \cap C_{SI}(\mathbb{R}^d)$ *and* $b_\varepsilon,\ \sigma \in C_b^{1,3}([0,T] \times \mathbb{R}^d)$. *Then we have*

$$\left| E\left[f\left(X_T^\varepsilon \right) \right] - E\left[f\left(\bar{X}_T^\varepsilon \right) \right] \right| \leq \frac{C}{n} \|b_\varepsilon\|_{3,\infty},$$

where C *is a positive constant and* $\|b_\varepsilon\|_{3,\infty}$ *is defined as follows:*

$$\|b_\varepsilon\|_{3,\infty} = \sum_{j=0}^{3} \left\| b_\varepsilon^{(j)} \right\|_\infty .$$

Note that we consider the Euler-Maruyama approximation with $\Delta t = \frac{T}{n}$.

Proof of Proposition 3 (1)

Proof.

- Define

$$\hat{Z}_t^{b_\varepsilon} = \int_0^t b_\varepsilon^* \sigma^{-1}(s, Y_s) dW_s - \frac{1}{2}\int_0^t b_\varepsilon^* a^{-1} b_\varepsilon(s, Y_s) ds,$$
$$\tilde{Z}_t^{b_\varepsilon} = \int_0^t b_\varepsilon^* \sigma^{-1}(\phi(s), \bar{Y}_{\phi(s)}) dW_s - \frac{1}{2}\int_0^t b_\varepsilon^* a^{-1} b_\varepsilon(\phi(s), \bar{Y}_{\phi(s)}) ds,$$

where W_t is a Brownian motion and

$$Y_t = x + \int_0^t \sigma(s, Y_s) dW_s, \quad \bar{Y}_t = x + \int_0^t \sigma\left(\phi(s), \bar{Y}_{\phi(s)}\right) dW_s.$$

- $\hat{\mathcal{L}}^{b_\varepsilon}(Y_t) \sim \mathcal{L}(X_t^\varepsilon)$ and $\tilde{\mathcal{L}}^{b_\varepsilon}(\bar{Y}_t) \sim \mathcal{L}(\bar{X}_t^\varepsilon)$ hold.
- Set $g(y,z) = \exp(z) f(y)$. Then

$$E\left[f\left(X_T^\varepsilon\right)\right] - E\left[f\left(\bar{X}_T^\varepsilon\right)\right] = E\left[g\left(Y_T, \hat{Z}_T^{b_\varepsilon}\right)\right] - E\left[g\left(\bar{Y}_T, \tilde{Z}_T^{b_\varepsilon}\right)\right].$$

Proof of Proposition 3 (2)

By using Taylor expansion,

$$= E\left[\nabla g\left(\theta Y_T + (1-\theta)\bar{Y}_T, \theta \hat{Z}_T^{b_\varepsilon} + (1-\theta)\tilde{Z}_T^{b_\varepsilon}\right)\left(Y_T - \bar{Y}_T\right)\right] \tag{3}$$

$$+ E\left[\partial_z g\left(\theta' Y_T + (1-\theta')\bar{Y}_T, \theta' \hat{Z}_T^{b_\varepsilon} + (1-\theta')\tilde{Z}_T^{b_\varepsilon}\right)\left(\hat{Z}_T^{b_\varepsilon} - \tilde{Z}_T^{b_\varepsilon}\right)\right],$$

where $\theta,\ \theta' \sim U(0,1)$ are independent of each other.

- $\mathcal{E}_t = Y_t - \bar{Y}_t$ is written as follows:

$$\mathcal{E}_t = \int_0^t \alpha_s \mathcal{E}_s dW_s + \int_0^t G_s dW_s$$

$$\Rightarrow \quad \mathcal{E}_t = U_t \int_0^t U_s^{-1} G_s dW_s - U_t \int_0^t U_s^{-1} \alpha(s) G_s ds,$$

where set $G_s = \sigma(s, \bar{Y}_s) - \sigma(\phi(s), \bar{Y}_{\phi(s)})$ and

$$U_t = 1 + \int_0^t \alpha_s U_s dW_s, \quad \alpha_s = \int_0^1 \partial_x \sigma(s, \xi Y_s + (1-\xi)\bar{Y}_s) d\xi.$$

Proof of Proposition 3 (3)

- Consider the first term of (3). (the rests are similar)

$$E\left[\underbrace{\nabla g\left(\theta Y_T + (1-\theta)\bar{Y}_T, \theta\hat{Z}_T^{b_\varepsilon} + (1-\theta)\tilde{Z}_T^{b_\varepsilon}\right)}_{=F}\mathcal{E}_T\right]$$

$$= E\left[F\left\{U_T\int_0^T U_s^{-1}G_s dW_s - U_T\int_0^T U_s^{-1}\alpha(s)G_s ds\right\}\right].$$

- By Taylor expansion, G_s is written as

$$G_s = \int_{\phi(s)}^{s}\partial_t\sigma\left(u,\bar{Y}_s\right)du \quad \left(\rightarrow \leq \frac{M}{n}\right)$$

$$+\int_0^1 \nabla\sigma\left(\phi(s),\beta\bar{Y}_s + (1-\beta)\bar{Y}_{\phi(s)}\right)\cdot\sigma\left(\phi(s),\bar{Y}_{\phi(s)}\right)\int_{\phi(s)}^{s} dW_u d\beta.$$

- By using the dual formula in the Malliavin calculus to the part $\int_{\phi(s)}^{s} dW_u$, the second term also has $\frac{M}{n}$. □

Lemmas

Lemma

For $p > 2 \vee d$, there exists some positive constant $C_1(p, \Lambda, T)$ such that

$$\left| E\left[f\left(X_T\right)\right] - E\left[f\left(X_T^{\varepsilon}\right)\right]\right| \leq C_1(p, \Lambda, T)\sqrt{\mathrm{Var}(f(x + B_T))}\|\mathbf{b} - \mathbf{b}_{\varepsilon}\|_{\mathbf{L^p}}.$$

The proof is the similar to the following lemma. □

Lemma

For $p > 2$, there exists some positive constant $C_2(p, \Lambda, T)$ such that

$$\left| E\left[f\left(\bar{X}_T\right)\right] - E\left[f\left(\bar{X}_T^{\varepsilon}\right)\right]\right| \leq C_2(p, \Lambda, T)\sqrt{\mathrm{Var}(f(x + B_T))}\|\mathbf{b} - \mathbf{b}_{\varepsilon}\|_{\mathbf{L^p}}.$$

Proof of Second Lemma (1)

Proof.

- Set $\beta(s) = b(x + B_{\phi(s)})$, $\beta_\varepsilon(s) = b_\varepsilon(x + B_{\phi(s)})$ and define

$$Z_t = 1 + \int_0^t Z_s \beta(s) dB_s, \quad Z_t^\varepsilon = 1 + \int_0^t Z_s^\varepsilon \beta_\varepsilon(s) dB_s.$$

- $Z_t - Z_t^\varepsilon$ is written as follows:

$$Z_t - Z_t^\varepsilon = \int_0^t (Z_s - Z_s^\varepsilon)\beta(s) dB_s + \int_0^t Z_s^\varepsilon(\beta(s) - \beta_\varepsilon(s)) dB_s.$$

- Then from $(a+b)^2 \leq 2a^2 + 2b^2$, we have

$$\Delta_t = E\left[\left|Z_t - Z_t^\varepsilon\right|^2\right] \leq 2\Lambda^2 \int_0^t \Delta_s ds + 2E\left[\int_0^t Z_s^\varepsilon(\beta(s) - \beta_\varepsilon(s))^2 ds\right].$$

- From (2), the second term in the RHS satisfies the following: ($C(p, \Lambda, T)$ is some positive constant)

$$E\left[\int_0^T Z_s^\varepsilon(\beta(s) - \beta_\varepsilon(s))^2 ds\right] \leq C(p, \Lambda, T) E\left[\int_0^T |\beta(s) - \beta_\varepsilon(s)|^p ds\right]^{\frac{2}{p}}.$$

Proof of Second Lemma (2)

- By the Gronwall's inequality,

$$\Delta_T \le C(p,\Lambda,T)e^{2\Lambda^2 T} E\left[\int_0^T |\beta(s)-\beta_\varepsilon(s)|^p ds\right]^{\frac{2}{p}}.$$

- By using the similar argument to Proposition 1, we have: ($\alpha, p > 2$、$\frac{1}{\alpha}+\frac{1}{p}<\frac{1}{2}$)

$$\left|E\left[f\left(\bar{X}_T\right)\right]-E\left[f\left(\bar{X}_T^{\varepsilon}\right)\right]\right|$$
$$=\left|E\left[Z_T f\left(x+B_T\right)\right]-E\left[Z_T^{\varepsilon} f\left(x+B_T\right)\right]\right|$$
$$\le C(\alpha,p,\Lambda,T)E\left[\int_0^T |\beta(s)-\beta_\varepsilon(s)|^p ds\right]^{\frac{1}{p}} \sqrt{\mathrm{Var}(f(x+B_T))}.$$

- By using an upper Gaussian estimation to the middle term, we have: ($\gamma > 1$)

$$E\left[\int_0^T \left|(b-b_\varepsilon)(x+B_{\phi(s)})\right|^p ds\right] \le C'(T,\gamma)\|b-b_\varepsilon\|^p_{L^{\frac{p\gamma}{\gamma-1}}}$$ □

Theorem 2

Theorem

Let b be a bounded measurable function on $\mathbb{R}^d$ which is Lipschitz except on a set G such that the Lebesgue measure $\mathrm{meas}(G(\varepsilon)) = O(\varepsilon^d)$. Then for any $f \in C_p^3(\mathbb{R}^d)$ and $p > d \vee 2$, we have

$$\left| E\left[f(X_T)\right] - E\left[f\left(\bar{X}_T\right)\right]\right| = O\left(n^{-\frac{d}{p+d}}\right).$$

Proposition

(Theorem 1 in Mackevičius [4]) Let b_ε be a bounded and Lipschitz continuous function with constant $\mathrm{Lip}(b_\varepsilon)$ and f be in $C_p^3(\mathbb{R}^d)$. Then there exists some positive constant $C(T, \Lambda, f)$ such that

$$\left| E\left[f\left(X_T^\varepsilon\right)\right] - E\left[f\left(\bar{X}_T^\varepsilon\right)\right]\right| \leq \frac{C(T,\Lambda,f)}{n}\mathrm{Lip}(b_\varepsilon).$$

Proof of Theorem 2

Proof.

- From the previous Proposition and two lemmas, for $p > 2$,

$$\left|E\left[f(X_T)\right] - E\left[f\left(\bar{X}_T\right)\right]\right|$$

$$\leq \left|E\left[f(X_T)\right] - E\left[f\left(X_T^{\varepsilon}\right)\right]\right| + \left|E\left[f\left(X_T^{\varepsilon}\right)\right] - E\left[f\left(\bar{X}_T^{\varepsilon}\right)\right]\right| + \left|E\left[f\left(\bar{X}_T^{\varepsilon}\right)\right] - E\left[f\left(\bar{X}_T\right)\right]\right|$$

$$\leq C(p, \Lambda, T)\sqrt{\mathrm{Var}(f(x + B_T))}\|b - b_\varepsilon\|_{L^p} + \frac{C(\Lambda, T, f)}{n}\mathrm{Lip}(b_\varepsilon).$$

- From $\mathrm{Lip}(b_\varepsilon) = O(\frac{1}{\varepsilon})$ and $\|b - b_\varepsilon\|_{L^p} = O(\varepsilon^{\frac{d}{p}})$, we have

$$\left|E\left[f(X_T)\right] - E\left[f\left(\bar{X}_T\right)\right]\right| \leq \frac{C}{n\varepsilon} + C'\varepsilon^{\frac{d}{p}}.$$

- An optimal choice of ε is $\varepsilon = O(n^{-\frac{p}{d+p}})$ and we obtain our consequence. □

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